SEGD-BCI Theoretical Framework and Mathematical Form of Self-Entanglement Bellman Equation for Decoding

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1 2.1 SEGD-BCI Theoretical Framework

The core bottleneck of Brain-Computer Interface (BCI) systems lies in the fact that traditional signal processing methods simplify neural electrical signals into discrete time-series vectors in Euclidean space, which breaks the inherent spatiotemporal coupling, nonlinear topological correlation, and dynamic self-organization characteristics of brain neural activities. It is difficult to achieve high-robustness and high-precision cross-scale neural intention decoding and closed-loop regulation. To solve this problem, this study introduces the original theoretical system of Self-Entangled Geometric Dynamics (SEGD), combined with the self-entanglement Bellman optimal decision criterion, to construct a brand-new SEGD-BCI theoretical framework. It models neural signal decoding and regulation as self-entanglement geometric spectrum analysis and dynamic optimal decision-making processes on high-dimensional manifolds, realizing the accurate characterization and engineering application of the essential laws of brain neural activities.

The core premise of the SEGD-BCI theoretical framework is to define the brain nervous system as a high-dimensional self-entanglement geometric dynamic system: brain neuron clusters and neural circuits form coupled sub-manifolds. Neural electrical activities (EEG/ECoG/neuronal spike signals) are not isolated time-series data, but the geometric evolution trajectories of the dynamic system on Riemannian manifolds. Their characteristics such as spatiotemporal distribution, phase coherence, and amplitude modulation are essentially external representations of the system's self-entangled state. The generation and transformation of neural intentions correspond to topological phase transitions, stable attractor transitions, and geometric phase evolution of the self-entanglement geometric system. The spatiotemporal correlation of multi-channel neural signals can be equivalent to the self-entanglement coupling effect inside the system, with non-local correlation characteristics, belonging to the self-organized entanglement behavior under the classical neural dynamics framework.

At the level of signal representation, this framework proposes the theory of self-entanglement geometric spectral decomposition, expanding the time-frequency analysis and spectrum analysis of traditional BCI signals into high-dimensional geometric spectrum representation. Analogous to the dimensional decomposition of photon energy by optical spectra, the neural self-entanglement geometric spectrum decomposes the original neural signals into multi-dimensional characteristic parameters such as self-entanglement eigenmodes, geometric curvature, topological invariants, and phase coherence spectra through manifold differential geometry and fiber bundle theory. It realizes the cross-scale unified characterization of microscopic neuron discharges and macroscopic cluster oscillations of neural activities, effectively retaining the global topological information and nonlinear correlation of neural signals, breaking through the linearization and localization limitations of traditional methods such as Fourier transform and independent component analysis, and improving the anti-noise ability and feature discrimination of weak neural signals.

At the level of dynamic decoding and optimal decision-making, the self-entanglement Bellman equation is introduced as the core control rule of SEGD-BCI, integrating the Bellman optimality principle of reinforcement learning with self-entanglement geometric constraints to construct a closed-loop optimal decoding model. This model takes the geometric properties of neural manifolds and self-entangled states as core state variables, taking into account decoding accuracy, signal-to-noise ratio, and manifold geometric conservation, realizing real-time dynamic optimal decoding of neural signals. It can adaptively track the dynamic evolution of neural manifolds and adapt to the non-stationarity and plasticity of brain neural activities. From the perspective of technical paradigm, the SEGD-BCI theoretical framework breaks the single mode of traditional BCI "signal acquisition-linear feature extraction-static classification", forming a full-process paradigm of "self-entanglement geometric state perception-geometric spectrum feature analysis-self-entanglement Bellman optimal decoding-closed-loop geometric regulation", providing a new theoretical path for solving key technical problems in the BCI field.

2 2.2 Mathematical Form of Self-Entanglement Bellman Equation in BCI Decoding

2.1 2.2.1 Core Symbol Definitions

To unify the theoretical framework and mathematical expressions, the core mathematical symbols for SEGD-BCI decoding are defined as follows:

1. Neural manifold and self-entangled state: The activity space of brain neural clusters is an n-dimensional Riemannian manifold $\mathcal{M}$ with metric tensor $g_{ij}(x)$. A point $x \in \mathcal{M}$ on the manifold corresponds to the instantaneous state of the neural cluster. The self-entangled state density operator $\hat{\rho}(x,t) \in \mathcal{B}(H)$ (H is the entanglement Hilbert space), and its classical probability distribution $\rho(x,t) = \mathrm{Tr}[\hat{\rho}(x,t)]$. The self-entanglement coupling operator $\hat{H}_{\mathrm{se}} = \hat{H}_0 + \hat{V}_{\mathrm{se}}$, where $\hat{H}_0$ is the basic neural Hamiltonian, $\hat{V}_{\mathrm{se}}(x,t)$ is the self-entanglement coupling potential satisfying Hermiticity $\hat{V}_{\mathrm{se}}^{\dagger} = \hat{V}_{\mathrm{se}}$.

2. Decoding state and action: System state at time t: $s_t = (\rho(x,t), \nabla g_{ij}(x))$, decoding action space $\mathcal{A}$, action $a_t \in \mathcal{A}$; intention space $\mathcal{I}$, true intention $i^* \in \mathcal{I}$.

3. Core parameters: Discount factor $\gamma \in [0,1]$, geometric constraint weight $\lambda > 0$, Riemannian manifold volume element $dV_g = \sqrt{\det(g_{ij})}dx$, geometric covariant derivative ∇_{geo}.

2.2 2.2.2 Reward Function Construction

The SEGD-BCI decoding reward function takes into account decoding accuracy, signal quality, and geometric topology conservation, with the expression:

$$r(s_t, a_t, i^*) = r_{\mathrm{acc}} + r_{\mathrm{snr}} + r_{\mathrm{geo}}$$

where the decoding accuracy reward $r_{\mathrm{acc}} = \mathbb{I}(a_t^{\mathrm{map}} = i^*)$ ($\mathbb{I}$ is the indicator function, taking 1 for correct decoding and 0 otherwise); the signal-to-noise ratio reward $r_{\mathrm{snr}} = \log_{10}(\mathrm{SNR}(x,t)/\mathrm{SNR}_0)$ (SNR_0 is the baseline SNR); the geometric conservation reward $r_{\mathrm{geo}} = -\lambda \cdot |\nabla_{\mathrm{geo}} a_t - \mathrm{id}|_g^2$ (id is the identity mapping), constraining the decoding process from destroying the topological structure of the neural manifold.

2.3 2.2.3 Self-Entanglement Bellman Optimality Equation

Define the self-entanglement geometric value function $V(s_t)$ as the cumulative expected reward of the optimal decoding strategy under state s_t, namely:

$$V(s_t) = \mathbb{E}_{\pi}\left[\sum_{k=t}^{\infty} \gamma^{k-t} r(s_k, a_k, i^*) \,\middle|\, s_t\right]$$

Based on Riemannian manifold constraints and self-entanglement coupling characteristics, the continuous-time self-entanglement Bellman optimality equation is derived:

$$V(\psi,t) = \max_{a \in \mathcal{A}}\left[r(\psi, a, i^*) + \left\langle \psi \,\middle|\, \hat{H}_{\mathrm{se}} + \nabla_{\mathrm{geo}} \,\middle|\, V(\psi, t+dt) \right\rangle\right]$$

where $\psi = (\rho(x,t), \beta(x,t))$ is the complete self-entanglement geometric state, $\langle \cdot | \cdot | \cdot \rangle$ is the joint inner product of Hilbert space and Riemannian manifold, and ψ is the evolved self-entanglement geometric state at time $t + dt$.

For the engineering requirements of real-time BCI decoding, the continuous equation is discretized to obtain an engineering-feasible discrete form:

$$V_t = \max_{a_t \in \mathcal{A}}\left[r_t + \gamma\left(\langle \rho_t | \hat{H}_{\mathrm{se},t} | \rho_{t+1} \rangle - g_{ij,t} \nabla V_{t+1}\right)\right]$$

where $V_t = V(s_t)$ is the value function at time t, $r_t = r(s_t, a_t, i^*)$ is the immediate reward, $\hat{H}_{\text{se},t}$ is the discretized self-entanglement coupling operator at time t, and ∇V_{t+1} is the discrete approximation of the geometric gradient of the value function.

2.4 2.2.4 Core Constraint Conditions

1. Self-entanglement conservation constraint: $\frac{d}{dt}\text{Tr}[\hat{\rho}(x,t)] = 0$
2. Geometric topological constraint: $\text{Ric}(g_{ij}) \geq \kappa$ (Ric is the Ricci curvature, κ is the lower bound of curvature)
3. Causality constraint: a_t only depends on historical states $\{s_0, s_1, \ldots, s_t\}$

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