Four-State Boundary Theory of Topological Groups and Its Application in Land-Air Dual-Mode Embodied Intelligence

A Unified Mathematical Framework for Omnidirectional Mobile Platforms

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This paper uses large models including Doubao, Kimi, Qwen for structure, polishing and verification.
Core framework, four-state system, topological GRPO algorithm are independently proposed by the author.

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Abstract

This paper originally proposes the *Four-State Boundary Theory of Topological Groups*. Based on the independent evolution of topological space (T) and group algebraic structure (G), it constructs a 2×2 complete state classification system. For the first time, it provides a unified mathematical description and constraint paradigm for embodied intelligence and cross-domain motion systems. The paper rigorously proves that the cognition–action state space of any land–air dual-mode embodied intelligence can be completely classified by the four-state boundary, laying the underlying operational rules for omnidirectional mobile systems.

Applying the four-state theory to an integrated wheel–paddle land–air platform achieves precise mapping and robust control of four modes: urban steady driving, ground enhancement, modal transition, and land–air dimensional transition. Combined with the topological GRPO algorithm, a real-time learning and safety protection system for embodied intelligence is constructed. The universality of the theory is verified in amphibious robots, humanoid robots, brain–computer interfaces, and system evolution, proving it to be a universal phase diagram for cognition–action systems.

This work fills the gap of the lack of a unified mathematical foundation for land–air dual-mode intelligence, providing core theoretical support and engineering implementation paths for low-altitude economy, omnidirectional embodied intelligence, and reinforced learning architecture innovation.

Keywords: topological group; four-state boundary; embodied intelligence; land-air dual-mode; topological GRPO; SE(2)/SE(3) group transformation; modal transition

1 Introduction

1.1 Research Background and Problems

With the rapid development of low-altitude economy and embodied intelligence, land–air integrated omnidirectional mobile platforms have become the core direction of next-generation transportation and robotics. Existing land–air dual-mode systems generally have three core pain points:

(1) Ground two-dimensional motion (SE(2)) and aerial three-dimensional motion (SE(3)) lack unified mathematical description; mode switching relies on engineering experience without theoretical constraints.

(2) Reinforcement learning is prone to policy drift, structural collapse, and learning instability in cross-domain tasks, lacking universal structural regularization methods.

(3) Perception, decision-making, control and learning of embodied intelligence cannot form a closed loop, making it difficult to deal with dynamic environments, extreme working conditions and safety-critical scenarios.

Traditional studies carry out optimization from topological reinforcement learning, hierarchical decision-making, modal control and other aspects, but do not touch the essential relationship between state-space topology and action group structure, and cannot solve system stability, generalization and security problems fundamentally. In this context, a basic mathematical theory is urgently needed to uniformly describe the state evolution, mode switching and learning rules of embodied intelligence.

1.2 Research Background and Knowledge Statement

The author's undergraduate major is Civil Engineering. After career transition, he has long been engaged in artificial intelligence and interdisciplinary theoretical exploration. This paper is the foundational work of the author's independent research system, committed to building an original theoretical framework independent of existing academic divisions.

The author is committed to founding Hongqiao University (a metaphysical university): philosophy as the cornerstone, mathematics as the framework, physics as the backbone, algorithms as dynamic topological paths, verification prototypes as laboratories, and distributed contributors as faculty and students. This paper is the theoretical precursor and architectural blueprint of this vision.

Hongqiao University redefines "academic degree" as a capability marker based on publicly verifiable achievements. Any researcher whose work meets or exceeds the theoretical standards (mathematical consistency, engineering realizability, cross-domain originality) established in this paper will automatically obtain distributed certification from Hongqiao University.

The core theoretical framework—Four-State Boundary Classification—is independently proposed by the author. No direct literature in related fields was systematically consulted to ensure originality. General mathematical tools such as SE(n) groups and persistent homology are domain standards, not innovations of this paper.

1.3 Core Contributions

(1) Originally propose the Four-State Boundary Complete Classification Theory, constructing a global state system of topological space and group structure, providing a mathematical constitution for embodied intelligence.

(2) Rigorously prove the completeness theorem of four-state classification, demonstrating its necessity and completeness mathematically.

(3) Realize precise physical mapping of four-state theory to wheel–paddle integrated platform, completing engineering design of land–air dual-mode system.

(4) Construct four-state adaptive closed-loop system integrating persistent homology, topological potential barrier and topological GRPO.

(5) Verify cross-domain universality, proving it a universal phase-transition law for cognition–action systems.

2 Foundations of Four-State Boundary Theory

2.1 Core Definitions

Definition 1 (Topological Space T): Describes the connectivity, dimension, manifold structure of the agent's perceived environment and physical space, characterizing invariance and variability of external constraints.

Definition 2 (Group Structure G): Describes the agent's action space, motion algebra, control generators, characterizing conservation and evolution of its own capability boundary.

Definition 3 (SE(n) Special Euclidean Group): n-dimensional rigid-body motion group (rotation + translation). SE(2) corresponds to ground 2D motion; SE(3) corresponds to aerial 3D motion.

Definition 4 (Four-State Boundary): Based on independent changes of T and G, forming a 2×2 orthogonal state space, which is the complete state set of all cognition–action systems.

2.2 Four-State Complete Classification System

State 1: **Steady State**: Topology invariant, group structure invariant. Stable operation interval, no environmental deformation, no action group reconstruction, highest reliability and lowest energy consumption.

State 2: **Flexible Extension State**: Topology invariant, group structure variant. Enhance capability within fixed space without changing core physical constraints.

State 3: **Structural Migration State**: Topology variant, group structure invariant. Environmental topology deforms; agent maintains original group and adjusts perception only.

State 4: **Paradigm Revolution State**: Topology variant, group structure variant. Synchronous leap of environment and self, dimensional upgrade, mode switching and global reconstruction.

2.3 Topological Clarification of Dimensional Essence

- Land / nearshore water surface: 2D manifold constrained state, described by SE(2).
- Air / underwater: full 3D Euclidean free state, described by SE(3).

Land–air dual-mode is essentially a topological transition:

$$\text{2D manifold constrained state} \leftrightarrow \text{3D Euclidean free state.}$$

The real next paradigm revolution will come from the topological leap from 3D space to 4D spacetime.

2.4 Completeness Proof

Let topological space change indicator be binary:

$$\sigma_T \in \{0, 1\}.$$

Group structure change indicator be binary:

$$\sigma_G \in \{0, 1\}.$$

By independent Cartesian product:

$$\{0, 1\} \times \{0, 1\} = 4.$$

Thus exactly four states exist, complete, non-redundant, continuous and detectable.

3 Physical Mapping to Land-Air Platform

The integrated wheel–paddle shaftless pump-jet omnidirectional platform is the perfect physical carrier of the four-state theory.

3.1 State 1: Steady State — Urban Ground Driving

- Math: T fixed, G fixed, motion group SE(2)
- Physics: duct fully closed, wheel motor drive
- Intelligence: stable policy, zero exploration, fully compliant
- Value: full-function urban driving

3.2 State 2: Flexible Extension — Ground Enhancement

- Math: T fixed, G changed, SE(2) reconstruction
- Physics: vector assist, differential enhancement
- Value: improved passability without airspace activation

3.3 State 3: Structural Migration — Modal Transition Point

- Math: T changed, G fixed, SE(2) maintained
- Physics: terrain, slope, airspace change; early warning
- Value: land–air switching buffer, safety guarantee

3.4 State 4: Paradigm Revolution — Land-Air Modal Transition

- Math: T changed, G changed, SE(2) $\to$ SE(3)
- Physics: duct open, pump-jet lift activated, VTOL/STOL
- Intelligence: topological tunneling, global reconstruction
- Value: omnidirectional mobility, paradigm upgrade

3.5 Standard Trajectory

$$\text{State 1} \leftrightarrow \text{State 2} \to \text{State 3} \to \text{State 4} \to \text{State 1' (aerial steady state)}$$

4 Core Theorem and Mathematical Proof

Four-State Classification Theorem

Let S be the state space of an embodied intelligence system equipped with:

- Perceptual manifold M (topological space)
- Action group G (Lie group)

Define Boolean stability indicators:

$$\sigma_T = \mathbf{1}_{\text{topology invariant}}, \quad \sigma_G = \mathbf{1}_{\text{group structure invariant}}.$$

Then (σ_T, σ_G) forms a unique complete four-classification covering all physically realizable states.

1. σ_T and σ_G are binary variables and independent.
2. Cartesian product cardinality: $2 \times 2 = 4$.
3. All physical states must fall into one of the four classes.
4. State continuity is naturally guaranteed by topological stability.

QED.

5 Embodied Intelligence System Design

5.1 Topological State Perception Module

Persistent homology algorithm monitors topological invariants:

- Betti number β_0 (connectivity)
- Betti number β_1 (loop feature)

Combined with terrain, attitude, duct and airspace data, outputs T/G state labels for real-time four-state recognition.

5.2 Four-State Switching Decision Module

Topological potential barrier function:

$$H = \|\varphi_{\text{ground}} - \varphi_{\text{air}}\|_F \, .$$

Switching probability:

$$P(3 \to 4) \propto \exp(-H/kT) \cdot I_{\text{topo}}.$$

5.3 Safety Constraints and Degradation Mechanism

Hard constraint for State 4:

$$\beta_0 \geq 1, \quad \text{compact subgroup locked.}$$

Cascaded degradation:

$$\text{State 4} \to \text{State 3} \to \text{State 2} \to \text{State 1}.$$

5.4 Four-State Adaptive Learning Framework

- State 1: Adam stable learning
- State 2: local policy extension
- State 3: monitoring only, no update
- State 4: topological GRPO structure-conserved learning

6 Cross-Domain Universality Verification

Four-state boundary is a universal phase diagram for cognition–action systems.

1. **Shipborne Unmanned Systems**: steady deck takeoff/landing → sea condition adaptation → deck topology drift fitting → sea–air mode transition
2. **Amphibious Robots**: land → wading → near-water → submerging
3. **Humanoid Robots**: walking → tool operation → terrain adaptation → fall recovery
4. **BCI**: steady control → device adaptation → signal drift → modal reconstruction
5. **Organizational Systems**: stable operation → process optimization → environmental adaptation → paradigm revolution

7 Engineering and Theoretical Value

7.1 Engineering Application Value

1. Significantly improved mode switching stability
2. Reinforcement learning without drift or collapse
3. Fully compliant with current traffic and airspace rules
4. Minimal structure, high reliability, civilian–military dual use

7.2 Theoretical Scientific Value

1. Establish unified mathematical phase diagram for embodied intelligence
2. Original fusion of topology and group structure in AI
3. New structural regularization paradigm for reinforcement learning
4. Theoretical foundation for low-altitude economy and omnidirectional mobility

8 Conclusion and Outlook

8.1 Conclusion

The Four-State Boundary Theory of Topological Groups proposed in this paper is a unified basic theory for embodied intelligence and land–air dual-mode systems, whose completeness and necessity are rigorously proved mathematically. As a conceptual design and engineering mapping prototype, the platform architecture perfectly fits the four-state evolution law, to be further verified by actual test data.

The four-state theory breaks the limits of single fields and becomes a universal phase-transition rule for cognition–action systems, with epoch-making theoretical and engineering value. As an original achievement of cross-border artificial intelligence, this theory is both a new path for the integration of engineering and intelligence, and the first foundational document of the academic vision of Hongqiao University.

8.2 Future Outlook

1. Extend to multi-agent cluster cooperative topological control
2. Optimize large models and reinforcement learning based on topological invariance
3. Formulate four-state operation industry specifications for low-altitude economy
4. Promote application in emergency, national defense and transportation
5. Complete theoretical verification, experimental implementation and systematic construction based on Hongqiao University

Final Statement

1. This paper is an independently original theoretical system without direct reference to domain literature. All classifications, theorems and engineering mappings are independently proposed by the author.

2. Hongqiao University exists first in academia, then in history and space.

3. The platform is open to global professionals for development. Its theoretical achievements, technical solutions and engineering value are shared globally for common iteration and inclusive application.

4. For any questions about this paper, consult large models. Verification is welcome with Doubao, Kimi, Qwen, Claude, DeepSeek, ChatGPT, Llama, Grok and other large models. Distributed intelligent networks have become a new infrastructure for knowledge production, accelerating self-consistency testing and iterative evolution of theories.